

Mathematical properties of the Navier-Stokes dynamical system for incompressible Newtonian fluids[§]

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Abstract

A remarkable feature of fluid dynamics is its relationship with classical dynamics and statistical mechanics. This has motivated in the past mathematical investigations concerning, in a special way, the "derivation" based on kinetic theory, and in particular the Boltzmann equation, of the incompressible Navier-Stokes equations (INSE). However, the connection determined in this way is usually merely asymptotic (i.e., it can be reached only for suitable limit functions) and therefore presents difficulties of its own. This feature has suggested the search of an alternative approach, based on the construction of a suitable inverse kinetic theory (IKT; Tessarotto et al., 2004-2008), which can avoid them. IKT, in fact, permits to achieve an exact representation of the fluid equations by identifying them with appropriate moment equations of a suitable (inverse) kinetic equation. The latter can be identified with a Liouville equation advancing in time a phase-space probability density function (PDF), in terms of which the complete set of fluid fields (prescribing the state of the fluid) are determined. In this paper we intend to investigate the mathematical properties of the underlying *finite-dimensional* phase-space classical dynamical system, denoted *Navier-Stokes dynamical system*, which can be established in this way. The result we intend to establish has fundamental implications both for the mathematical investigation of Navier-Stokes equations as well as for diverse consequences and applications in fluid dynamics and applied sciences.

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I. 1 - INTRODUCTION

A fundamental theoretical issue in mathematical physics is the search of a possible *finite-dimensional* classical dynamical system - to be denoted as *Navier-Stokes* (NS) *dynamical system* - which uniquely advances in time the complete set of fluid fields which characterize a fluid system. In the case of an incompressible isothermal Newtonian fluid (also known as *NS fluid*) these are identified with the (mass) fluid velocity and the (non-negative) scalar fluid pressure, which in the domain of the fluid itself satisfy the *incompressible NS equations* (INSE). The reason why the determination of such a dynamical system is so important is that its existence is actually instrumental for the establishment of theorems of existence and uniqueness for the related initial-boundary value problem (*INSE problem*). In this paper we prove that, based on the inverse kinetic theory approach (IKT) developed by Tessarotto *et al.* (2004-2007 [1, 2, 5]), the problem can actually be given a well-defined formulation. Main goal of the paper is the establishment of an *equivalence theorem* between the INSE problem and the NS dynamical system. Basic consequences and applications of this result are pointed out. In particular, contrary to the widespread view according to which the phase-space dynamical system characterizing the fluid fields should be infinite dimensional, here we intend to prove that a *finite-dimensional classical dynamical system* exists which advances in time the complete set of fluid fields. This is realized by the NS dynamical system.

II. 2 - THE STRONG STOCHASTIC INSE PROBLEM

For definiteness, let us assume that the complete set of fluid fields $\{Z\} \equiv \{\rho, \mathbf{V}, p\}$, respectively denoting the mass density, the fluid velocity and the fluid pressure, describing an NS fluid, are local strong solutions of the equations

$$\rho = \rho_o, \tag{1}$$

$$\nabla \cdot \mathbf{V} = 0, \tag{2}$$

$$N\mathbf{V} = 0, \tag{3}$$

$$Z(\mathbf{r}, t_o) = Z_o(\mathbf{r}), \tag{4}$$

$$Z(\mathbf{r}, t)|_{\partial\Omega} = Z_w(\mathbf{r}, t)|_{\partial\Omega}, \tag{5}$$

In particular, Eqs. (1)-(5) are respectively the *incompressibility, isochoricity and Navier-Stokes equations* and the initial and Dirichlet boundary conditions for $\{Z\}$, with $\{Z_o(\mathbf{r})\}$ and $\{Z_w(\cdot, t)|_{\partial\Omega}\}$ suitably prescribed initial and boundary-value fluid fields, defined respectively at the initial time $t = t_o$ and on the boundary $\partial\Omega$. In the remainder, for definiteness, we shall require that:

1. Ω coincides with the Euclidean space E^3 on \mathbb{R}^3 (*external domain*) and $\partial\Omega$ with the improper plane of \mathbb{R}^3 ;
2. I is identified with the real axis \mathbb{R} (*global domain*).

We shall assume that the fluid fields are continuous in $\bar{\Omega} \times I$ and fulfill the inequalities

$$p > 0, \quad (6)$$

$$\rho > 0. \quad (7)$$

Here the notation as follows. N is the *NS nonlinear operator*

$$N\mathbf{V} = \frac{D}{Dt}\mathbf{V} - \mathbf{F}_H, \quad (8)$$

with $\frac{D}{Dt}\mathbf{V}$ and \mathbf{F}_H denoting respectively the *Lagrangian fluid acceleration* and the *total force per unit mass*

$$\frac{D}{Dt}\mathbf{V} = \frac{\partial}{\partial t}\mathbf{V} + \mathbf{V} \cdot \nabla \mathbf{V}, \quad (9)$$

$$\mathbf{F}_H \equiv -\frac{1}{\rho_o}\nabla p + \frac{1}{\rho_o}\mathbf{f} + \nu \nabla^2 \mathbf{V}, \quad (10)$$

while ρ_o and $\nu > 0$ are the constant *mass density* and the constant *kinematic viscosity*. In particular, \mathbf{f} is the *volume force density* acting on the fluid, namely which is assumed of the form

$$\mathbf{f} = -\nabla\phi(\mathbf{r}, t) + \mathbf{f}_R, \quad (11)$$

$\phi(\mathbf{r}, t)$ being a suitable scalar potential, so that the first two force terms [in Eq.(10)] can be represented as $-\nabla p + \mathbf{f} = -\nabla p_r + \mathbf{f}_R$, with

$$p_r(\mathbf{r}, t) = p(\mathbf{r}, t) - \phi(\mathbf{r}, t), \quad (12)$$

denoting the *reduced fluid pressure*. As a consequence of Eqs.(1) and (2) it follows that the fluid pressure necessarily satisfies the *Poisson equation*

$$\nabla^2 p = S, \quad (13)$$

where the source term S reads

$$S = -\rho_o \nabla \cdot (\mathbf{V} \cdot \nabla \mathbf{V}) + \nabla \cdot \mathbf{f}. \quad (14)$$

Here we shall assume, furthermore, that the fluid fields $\{Z\}$, together with the volume force density \mathbf{f} and the initial and boundary conditions $\{Z_o(\mathbf{r})\}$, $\{Z_w(\mathbf{r}, t)|_{\partial\Omega}\}$ are all stochastic functions (see Appendix) of the form

$$\begin{aligned} Z &= Z(\mathbf{r}, t, \alpha), \\ \mathbf{f} &= \mathbf{f}(\mathbf{r}, t, \alpha) \\ Z_o &= Z_o(\mathbf{r}, \alpha) \\ Z_w &= Z_w(\mathbf{r}, t)|_{\partial\Omega} \end{aligned} \quad (15)$$

where $\alpha \in V_\alpha$ are stochastic variables assumed independent of (\mathbf{r}, t) . Eqs.(1)-(5) then denote the initial-boundary value problem for the stochastic incompressible Navier-Stokes equations (*strong stochastic INSE problem*).

III. 3 - THE IKT-STATISTICAL MODEL

A fundamental aspect of fluid dynamics lays in the construction of statistical models for the fluid equations [16]. In this connection a possible viewpoint is represented by the construction of the so-called *IKT-statistical model* able to yield as moments of the PDF the *whole set of fluid fields* $\{Z\}$ which determine the fluid state and in which the same PDF satisfies a Liouville equation. Despite previous attempts (Vishik and Fursikov, 1988 [3] and Ruelle, 1989 [4]) the existence of such a dynamical system has remained for a long time an unsolved problem.

This type of approach has actually been achieved for incompressible NS fluids [1]. Its applications and extensions are wide-ranging and concern in particular: incompressible thermofluids [8], quantum hydrodynamic equations (see [6, 9]), phase-space Lagrangian dynamics [10], tracer-particle dynamics for thermofluids [11, 17], the evolution of the fluid pressure in incompressible fluids [12], turbulence theory in Navier-Stokes fluids [9, 13] and magnetofluids [15] and applications of IKT to lattice-Boltzmann methods [14].

The IKT-statistical model is based on the introduction of a PDF depending on the local configuration-space vector \mathbf{r} , $f_1(t) \equiv f_1(\mathbf{r}, \mathbf{v}, t, \alpha)$ (*1-point velocity PDF*) defined on the

phase-space $\Gamma_1 = \Omega \times U$ [with $U \equiv \mathbb{R}^3$] and identified with a strictly positive function such that the complete set of fluid fields $\{Z\}$ associate to the strong stochastic INSE problem can be represented in terms of the functionals (*velocity moments*)

$$\int_U d\mathbf{v} G f_1(t) = \{1, \mathbf{V}(\mathbf{r}, t, \alpha), p_1(\mathbf{r}, t, \alpha)\}, \quad (16)$$

(*Requirement #1; correspondence principle*). Here $G = \{1, \mathbf{v}, u^2/2\}$ while $p_1(\mathbf{r}, t, \alpha) > 0$ denotes the *kinetic pressure*

$$p_1(\mathbf{r}, t, \alpha) = p(\mathbf{r}, t, \alpha) + p_0(t, \alpha) - \phi(\mathbf{r}, t, \alpha), \quad (17)$$

with $p_0(t, \alpha) > 0$ (the *pseudo-pressure*) a strictly positive, smooth, i.e., at least $C^{(1)}(I)$, real function and $\phi(\mathbf{r}, t, \alpha)$ a suitably defined potential. In addition $f_1(t)$ is assumed to obey the Liouville equation - or inverse kinetic equation (IKE) according to the notation of Ref.[2] -

$$L(\mathbf{r}, \mathbf{v}, t; f_1(t)) f_1(t) = 0 \quad (18)$$

(*Requirement #2*) with $L(\mathbf{r}, \mathbf{v}, t; f_1(t))$ denoting the Liouville streaming operator $L(\mathbf{r}, \mathbf{v}, t; f_1(t)) \cdot \equiv \frac{\partial}{\partial t} \cdot + \frac{\partial}{\partial \mathbf{x}} \cdot \{\mathbf{X}(\mathbf{x}, t; f_1(t)) \cdot\}$ and $\mathbf{F}(f_1(t))$ a suitable smooth vector field defined in such a way that the moment equations of (18) obtained for $G = \{1, \mathbf{v}, \rho_o u^2/3\}$ [\mathbf{u} denoting the relative velocity $\mathbf{u} \equiv \mathbf{u}(\mathbf{r}, t, \alpha) = \mathbf{v} - \mathbf{V}(\mathbf{r}, t, \alpha)$] coincide respectively with Eqs.(2), (3) and again (2). This implies that the initial value problem associated to the vector field

$$\mathbf{X}(\mathbf{x}, t; f_1(t)) = \{\mathbf{v}, \mathbf{F}(f_1(t))\}, \quad (19)$$

$$\begin{cases} \frac{d\mathbf{x}}{dt} = \mathbf{X}(\mathbf{x}, t; f_1(t)), \\ \mathbf{x}(t_o) = \mathbf{x}_o \end{cases} \quad (20)$$

necessarily defines a dynamical system. In particular if $\mathbf{x}(t) = \chi(\mathbf{x}_o, t_o, t)$ is the general solution of (20), this is identified with the flow

$$T_{t_o, t} : \mathbf{x}_o \rightarrow \mathbf{x}(t) = T_{t_o, t} \mathbf{x}_o \equiv \chi(\mathbf{x}_o, t_o, t) \quad (21)$$

generated by $\mathbf{X}(\mathbf{x}, t; f_1(t))$. Furthermore it is assumed that Eq.(18) admits as a particular solution $f_1(t)$ the Gaussian PDF

$$f_M(\mathbf{x}, t, \alpha) = \frac{1}{\pi^{3/2} v_{th}^3(\mathbf{r}, t, \alpha)} \exp \left\{ -\frac{u^2}{v_{th}^2(\mathbf{r}, t, \alpha)} \right\}, \quad (22)$$

where $v_{th}^2(\mathbf{r}, t, \alpha) = 2p_1(\mathbf{r}, t, \alpha)/\rho_o$. More precisely it is assumed that $f_M(\mathbf{x}, t, \alpha)$ is a particular solution of Eq.(18) if and only if the fluid fields $Z(\mathbf{r}, t, \alpha)$ are solutions of the strong stochastic INSE problem (*Requirement #3*).

In the following we intend to investigate, in particular, the consequences of requirements (16), (18) and (22) for the problem posed in this paper.

IV. 4 - THE NAVIER-STOKES DYNAMICAL SYSTEM

According to a certain misconception, dynamical systems for continuous fluids cannot be finite dimensional due to the fact that the fluid equations are PDEs for the relevant set of fluid fields $\{Z\}$. However, it is easy to show that this is not the case even in the so-called Lagrangian description of fluid dynamics. For a NS fluid this is realized by parametrizing the fluid velocity \mathbf{V} in terms of the Lagrangian path (LP) $\mathbf{r}(t)$. In the present notation this is solution of the problem

$$\begin{cases} \frac{D\mathbf{r}(t)}{Dt} = \mathbf{V}(\mathbf{r}(t), t, \alpha), \\ \mathbf{r}(t_o) = \mathbf{r}_o, \end{cases} \quad (23)$$

where $\frac{D}{Dt}$ is the Lagrangian derivative (9). As a consequence the NS equation becomes

$$\frac{D\mathbf{V}(\mathbf{r}(t), t)}{Dt} = \mathbf{F}_H(\mathbf{r}(t), t), \quad (24)$$

which [with $\mathbf{F}_H(\mathbf{r}(t), t)$ considered prescribed] can be treated as an ODE and integrated along a LP yielding

$$\mathbf{V}(\mathbf{r}(t), t) = \mathbf{V}(\mathbf{r}_o, t_o) + \int_{t_o}^t dt' \mathbf{F}_H(\mathbf{r}(t'), t'). \quad (25)$$

This permits to determine, *for all* $\mathbf{r} \equiv \mathbf{r}(t)$ and $(\mathbf{r}_o, t_o) \in \Omega \times I$, the vector field $\mathbf{V}(\mathbf{r}, t)$. Therefore the *finite-dimensional dynamical system* $(\mathbf{r}_o, t_o) \rightarrow (\mathbf{r}(t), t)$ defined by Eq.(23) actually generates the time evolution of $\mathbf{V}(\mathbf{r}, t, \alpha)$. Nevertheless, in this description the fluid pressure is actually not directly determined [in fact this requires solving the Poisson equation (13)].

We intend to show that a dynamical system advancing in time the complete set of fluid fields for a NS fluid is realized by the dynamical system $T_{t_o, t}$ [NS dynamical system; see Eq.(21)]. Its precise definition depends manifestly on the vector field $\mathbf{F}(f_1(t))$. The task [of defining $\mathbf{F}(f_1(t))$] is achieved by the IKT approach developed in Ref.[2]. As a consequence

it follows that $\mathbf{F}(f_1(t))$ can be non-uniquely determined [2, 5] in terms of a smooth vector field which is at least $C^{(1)}(\Gamma_1 \times I \times V_\alpha)$.

In this case, as a fundamental mathematical result, we intend to prove here *the equivalence between the strong stochastic INSE problem and the NS dynamical system*. In other words the Liouville equation (18) can be shown to be equivalent to

$$f_1(\mathbf{x}, t, \alpha) = J(t, \alpha) f_1(\chi(\mathbf{x}, t, t_o, \alpha), t_o, \alpha), \quad (26)$$

where $\mathbf{x}(t) = T_{t_o, t} \mathbf{x}_o$ is the general solution of (20) for $\mathbf{F} \equiv \mathbf{F}(f_1(t))$ and where

$$J(t, \alpha) = \exp \left\{ \int_{t_o}^t dt' \frac{\partial}{\partial \mathbf{v}(\mathbf{t}', \alpha)} \cdot \mathbf{F}(\mathbf{x}(t', \alpha), t', \alpha; f_1(t')) \right\} \equiv \left| \frac{\partial \mathbf{x}(t, \alpha)}{\partial \mathbf{x}_o} \right| \quad (27)$$

is the Jacobian of the flow $T_{t_o, t}$ [see Eq.(21)]. Therefore, the NS dynamical system necessarily advances in time the PDF $f_M(\mathbf{x}, t_o, \alpha)$ so that it is *identically* a solution of the Liouville equation (18). The result can be established on general grounds, i.e., for an arbitrary vector field $\mathbf{F}((f_1(t)))$ fulfilling Requirements #1-#3. The following result holds:

THM.1 - Equivalence theorem

In validity of Requirements #1 – #3 the strong stochastic INSE problem is equivalent to the NS dynamical system (21) .

PROOF

The proof is immediate. In fact, if

$$\mathbf{x} \equiv \mathbf{x}(t, \alpha) = \chi(\mathbf{x}_o, t_o, t, \alpha) \quad (28)$$

is the solution of Eq.(20) (which is assumed to exist and define at least a $C^{(2)}$ –diffeomorphism, its inverse transformation is simply

$$\mathbf{x}_o = \chi(\mathbf{x}, t, t_o, \alpha). \quad (29)$$

Therefore by differentiating Eq.(26) it follows

$$\frac{d}{dt} f_1(\mathbf{x}, t, \alpha) - \frac{d}{dt} \{ J(t, \alpha) f_1(\chi(\mathbf{x}, t, t_o, \alpha), t_o, \alpha) \} = 0, \quad (30)$$

which recovers Eq.(26) and admits as a particular solution $f_M(t) \equiv f_M(\mathbf{x}, t, \alpha)$ when subject to the initial condition $f_M(\mathbf{x}_o, t_o, \alpha)$. Hence in terms of such an equation the NS dynamical

system advances in time *the complete set of fluid fields*. Therefore, the fluid velocity and the kinetic pressure at time t , i.e., $\mathbf{V}(t) \equiv \mathbf{V}(\mathbf{r}, t, \alpha)$ and $p_1(t) \equiv p_1(\mathbf{r}, t, \alpha)$, follow from the moment equations (16). Q.E.D.

V. 5. CONCLUSIONS

This work is motivated by the analogy between hydrodynamic description and the theory of classical dynamical systems. For greater generality the case of stochastic fluid equations has been considered. The problem of the equivalence between the initial-boundary value problem for incompressible Navier-Stokes equations and the Navier-Stokes dynamical system introduced in Ref.[2] has been investigated. Indeed, the theory here developed applies both to deterministic and stochastic fluid fields. In fact, in both cases the time evolution of f_1 is determined by a Liouville equation [see Eq.(18)] which evolves in time also the complete set of fluid fields (all represented in terms of moments of the same PDF). Contrary to the misconception according to which the phase-space dynamical system characterizing the fluid fields $\{Z\}$ of a continuous fluid system should be infinite dimensional, here we have proven that the finite-dimensional NS classical dynamical system advances in time *the complete set of fluid fields*, determined in terms of velocity moments of the 1-point PDF $f_1(\mathbf{x}, t, \alpha)$. The theory here developed applies generally to stochastic fluid equations. As shown elsewhere [13, 15, 16], this represents a convenient treatment for the statistical theory of turbulence, historically referred to the work of Kolmogorov (Kolmogorov, 1941 [18]) and Hopf (Hopf, 1950/51 [19]).

The theory has important consequences which concern fundamental aspects of fluid dynamics:

- determination of the NS dynamical system advancing in time the complete set of fluid fields of a turbulent NS fluid [16];
- construction of the initial conditions for the 1-point PDF f_1 [20];
- determination of the time-evolution of passive scalar and tensor fields [12];
- construction of the exact equations of motion for ideal tracer-particle dynamics in a turbulent NS fluids [17];

- construction of multi-point PDFs for turbulent NS fluids [21];
- statistical treatment of homogeneous, isotropic and stationary turbulence based on IKT [22].

VI. APPENDIX - STOCHASTIC VARIABLES

Let (S, Σ, P) be a probability space; a measurable function $\alpha : S \longrightarrow V_\alpha$, where $V_\alpha \subseteq \mathbb{R}^k$, is called *stochastic* (or *random*) *variable*.

A stochastic variable α is called *continuous* if it is endowed with a *stochastic model* $\{g_\alpha, V_\alpha\}$, namely a real function g_α (called as *stochastic PDF*) defined on the set V_α and such that:

- 1) g_α is measurable, non-negative, and of the form

$$g_\alpha = g_\alpha(\mathbf{r}, t, \cdot); \quad (31)$$

- 2) if $A \subseteq V_\alpha$ is an arbitrary Borelian subset of V_α (written $A \in \mathcal{B}(V_\alpha)$), the integral

$$P_\alpha(A) = \int_A d\mathbf{x} g_\alpha(\mathbf{r}, t, \mathbf{x}) \quad (32)$$

exists and is the probability that $\alpha \in A$; in particular, since $\alpha \in V_\alpha$, g_α admits the normalization

$$\int_{V_\alpha} d\mathbf{x} g_\alpha(\mathbf{r}, t, \mathbf{x}) = P_\alpha(V_\alpha) = 1. \quad (33)$$

The set function $P_\alpha : \mathcal{B}(V_\alpha) \rightarrow [0, 1]$ defined by (32) is a probability measure and is called distribution (or law) of α . Consequently, if a function $f : V_\alpha \longrightarrow V_f \subseteq \mathbb{R}^m$ is measurable, f is a stochastic variable too.

Finally define the *stochastic-averaging operator* $\langle \cdot \rangle_\alpha$ (see also [15]) as

$$\langle f \rangle_\alpha = \langle f(\mathbf{y}, \cdot) \rangle_\alpha \equiv \int_{V_\alpha} d\mathbf{x} g_\alpha(\mathbf{r}, t, \mathbf{x}) f(\mathbf{y}, \mathbf{x}), \quad (34)$$

for any P_α -integrable function $f(\mathbf{y}, \cdot) : V_\alpha \rightarrow \mathbb{R}$, where the vector \mathbf{y} is some parameter.

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